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Engineering Notes

Analysis of Periodic and Quasi-Periodic Orbits in the Earth–Moon System

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I. Introduction

ONE of the earliest investigations of the Poincaré surfaces of section method [1] to discover periodic orbits in the restricted three-body problem was by Henon [2–5] in a number of publications. Jefferys [6] also used this method to evolve a large number of periodic orbits in the restricted three-body problem. After that, this technique has been used by many researchers to find periodic orbits in the restricted three-body problem. Some of the important contributions are by Smith [7], Smith and Szebehely [8], Tuckness [9,10], Scott and Spencer [11], Howell and Kakoi [12], Demeyer and Gurfil [13], Guilera [14], and Kolemen et al. [15]. The motivation for the present study comes from the work of Winter [16], which deals with a family of simply symmetric retrograde periodic orbits around the moon in the rotating Earth–moon–particle system in the framework of the planar circular restricted three-body problem for the Earth–moon mass ratio (0.01215). This family of periodic orbits is one particular case of the family of periodic orbits classified by Broucke [17].

II. Planar Circular Restricted Three-Body Problem

In this model, two bodies (known as the primaries) of masses m_1 and m_2 are assumed to rotate about their center of mass in circular orbits in a plane under the influence of their mutual gravitational attraction [18,19]. The objective is to describe the motion of a third infinitesimal mass, typically representing a spacecraft. To simplify the analysis for this study, the infinitesimal mass is further restricted in the plane of motion of the two primaries. The resulting problem is commonly referred to as the planar circular restricted three-body problem.

The unit of length is the distance between the primaries and the unit of time is chosen in order to set the gravitational parameter as unity. We use a rotating coordinate system (x, y), which rotates uniformly with unit velocity, where the x axis is along the line joining the two primaries. In the present study, we simulate the motion of a spacecraft placed in the Earth–moon system. The coordinates are scaled so that the Earth is at the location $(-\mu, 0)$, and the moon is at $(1-\mu, 0)$, where μ is the mass ratio $m_2/(m_1+m_2)$, and m_1 and m_2 are the masses of Earth and moon, respectively. The equations for the motion of the spacecraft in the rotating frame of reference are

$$\ddot{x} - 2\dot{y} - x = -(1-\mu)\frac{(x+\mu)}{r_1^3} - \mu\frac{(x-1+\mu)}{r_2^3} \tag{1}$$

$$\ddot{y} + 2\dot{x} - y = -\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right)y \tag{2}$$

where

$$r_1^2 = (x+\mu)^2 + y^2, \qquad r_2^2 = (x-1+\mu)^2 + y^2 \tag{3}$$

The dynamical system has an integral of the motion, known as Jacobi's constant, given by

$$C = x^2 + y^2 + 2\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right) - \dot{x}^2 - \dot{y}^2 \tag{4}$$

III. Poincaré Surface of Section

The orbital elements of the test particle at any instant can be determined from its position (x, y) and velocity $(\dot{x}, \dot{y})$, which correspond to a point in a four-dimensional phase space. To visualize this four-dimensional manifold (which includes the periodic orbits, quasi-periodic orbits, and the nonperiodic trajectories on a two-dimensional figure), we need to constrain the manifold by two dimensions. A convenient way of achieving this is to choose the trajectories that have the same value of C, and to take a Poincaré section when these orbits cross a plane (say, $y = 0$). This technique is good at identifying periodic and quasi-periodic orbits, determining the regular or chaotic nature of the trajectory, and determining the extent of the particle's motion throughout the phase space. As per Kolmogorov–Arnold–Moser (KAM) theory, the point represents a periodic orbit in the rotating frame, and the closed curves around the point correspond to the quasi-periodic orbits.

IV. Numerical Results

As in Winter [16], the starting conditions in our computations were chosen such that for each value of the Jacobi constant C, the values of x were selected by taking $y = \dot{x} = 0$ and $\dot{y} > 0$. This meant that the integration was usually started at the pericenter of the particle's orbit. We have used the fourth-order Runge–Kutta–Gill method for numerical integration of the equations of motion. The integration step size of 0.01 in time was found to be adequate for our numerical studies. Over 1500 starting conditions in the range of Jacobi constant C between 1.5 and 4.24, which covers the region that contains the periodic orbits considered, were used. Experimentation for the distance interval Δx between two consecutive starting conditions with the same C was done for $\Delta x = 0.01$ to 0.0001. It was found that $\Delta x = 0.001$ is more suitable for our studies. In the present dynamical system, the largest of the quasi-periodic orbits (KAM tori), which correspond to the maximum amplitude oscillation, can be taken as a parameter to measure the degree of stability of the periodic orbit with respect to the region around it in the phase space.

With the aim to obtain periodic orbits near the moon, we have generated Poincaré surfaces of section up to $C = 4.24$. We divide this family of orbits into two categories. Category I is orbits consisting of oscillations near the moon, and category II is orbits consisting of oscillations near the Earth.

The evolution of the category I orbits, in terms of increasing value of the Jacobi constant, can be further divided into four distinct stages. In the first stage, from $C = 2.4$ to 2.84, the maximum amplitude of oscillation decreases uniformly until it disappears at $C = 2.84895$. Then, in the second stage, the islands reappear again at $C = 2.85$ and the stability grows, reaching a peak at $C = 2.9$ and then decreasing until they disappear completely around $C = 2.955$. The periodic orbit at $C = 2.954$ corresponds to the collinear Lagrangian equilibrium point L_1 ($x = 0.83689$). In the above two zones, our

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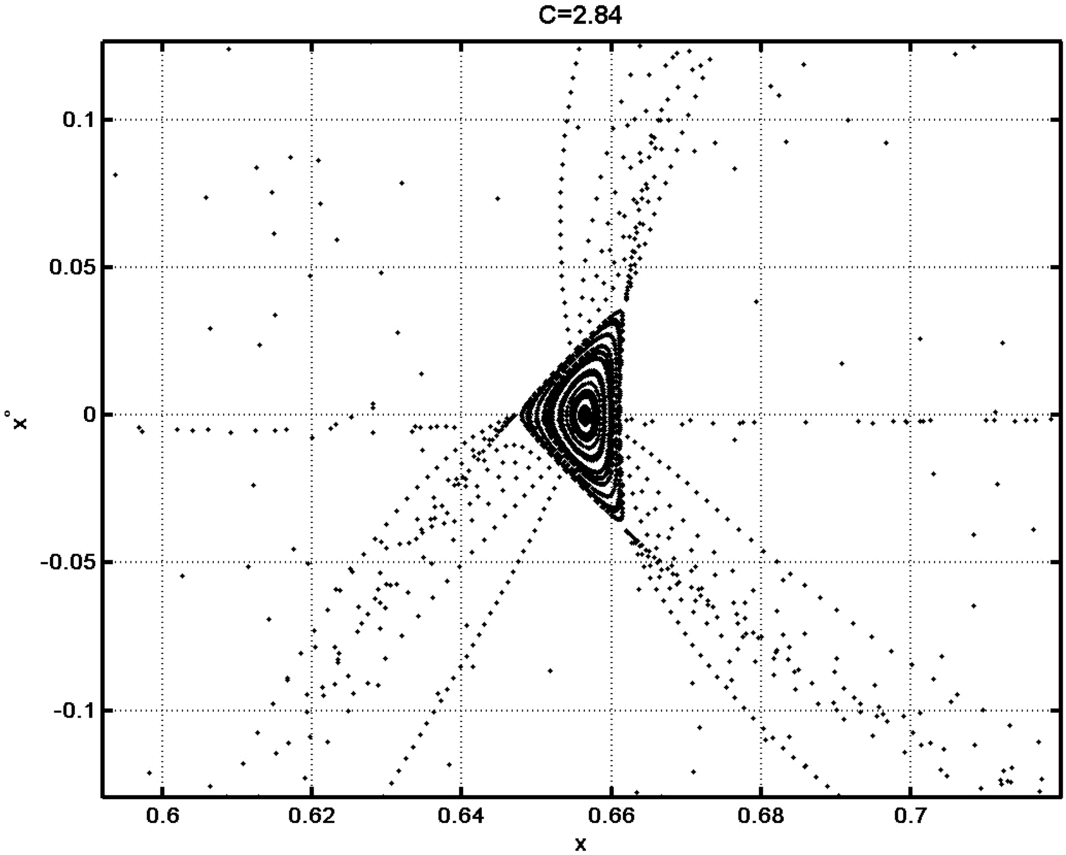

Fig. 1 Poincaré surface of section for Jacobi constant $C = 2.84$.

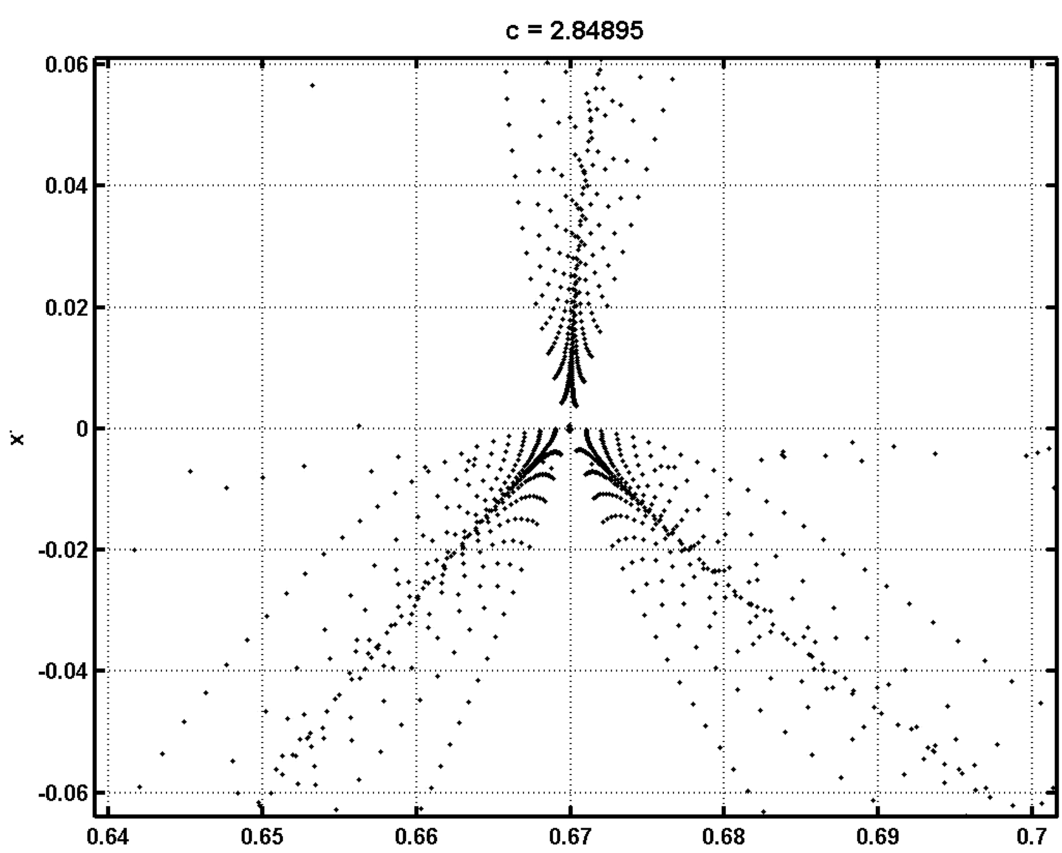

Fig. 2 Poincaré surface of section for Jacobi constant $C = 2.84895$.

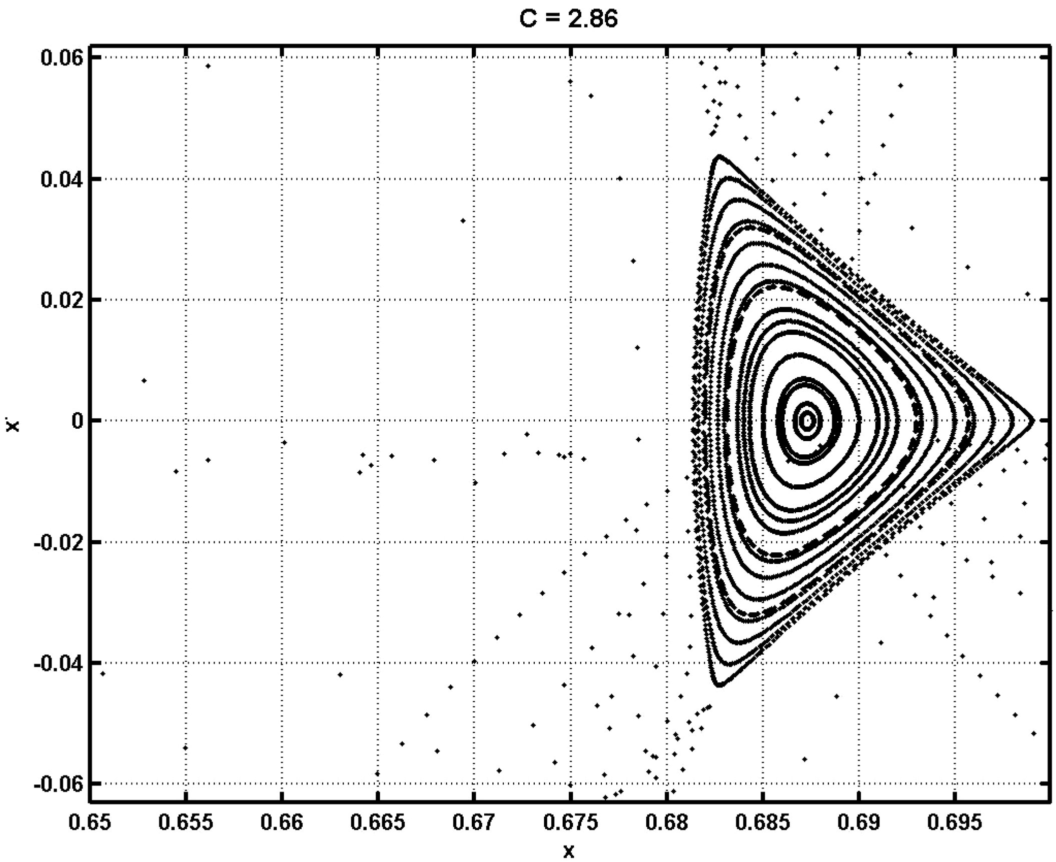

Fig. 3 Poincaré surface of section for Jacobi constant $C = 2.86$.

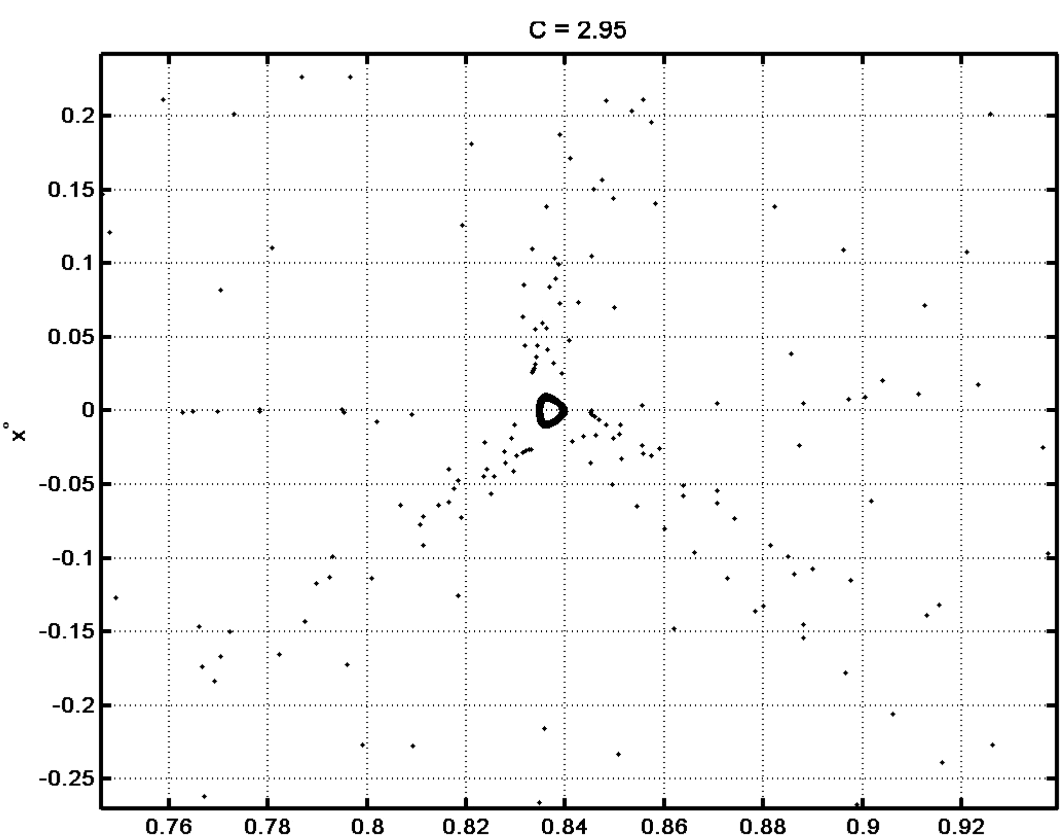

Fig. 4 Poincaré surface of section for Jacobi constant $C = 2.95$.

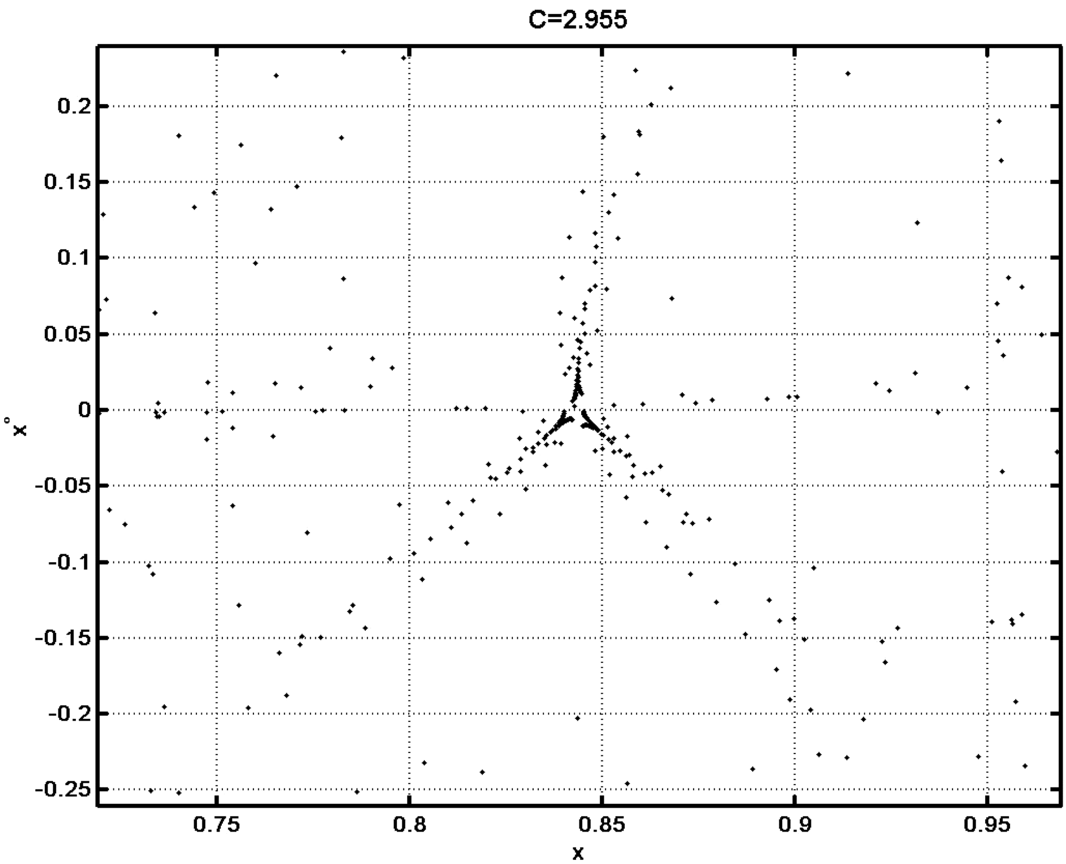

Fig. 5 Poincaré surface of section for Jacobi constant $C = 2.955$.

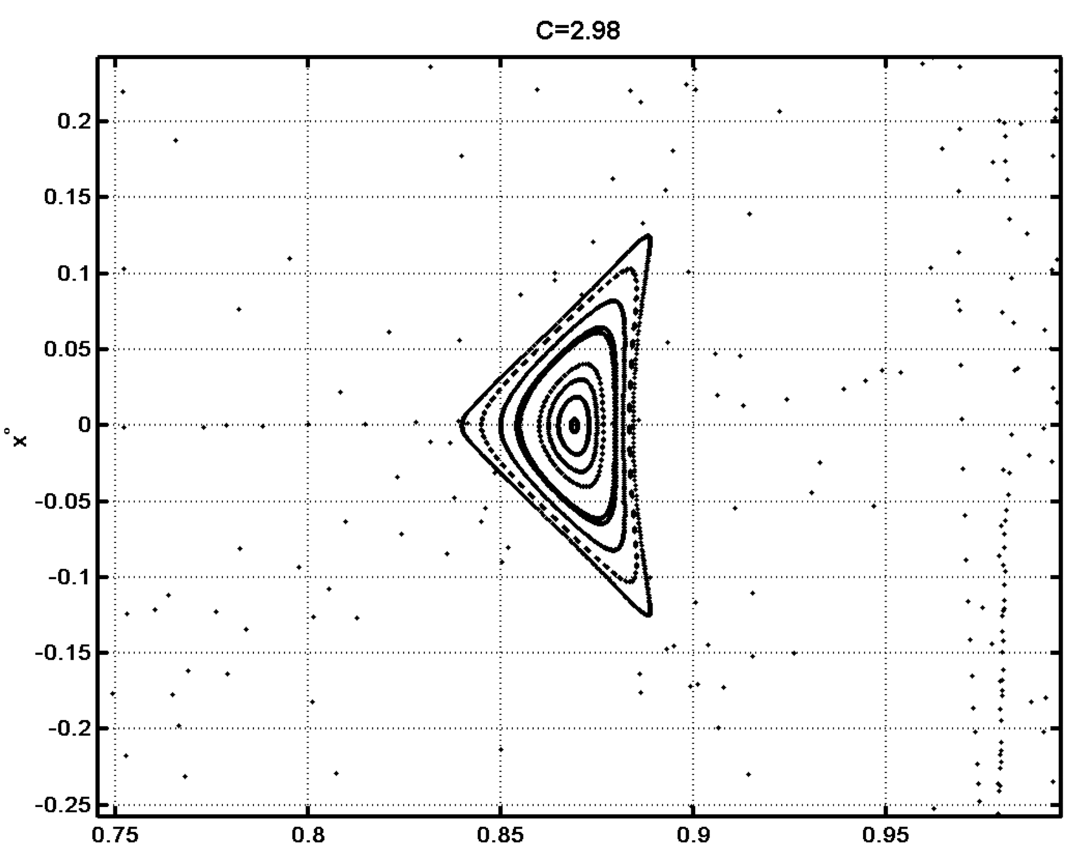

Fig. 6 Poincaré surface of section for Jacobi constant $C = 2.98$.

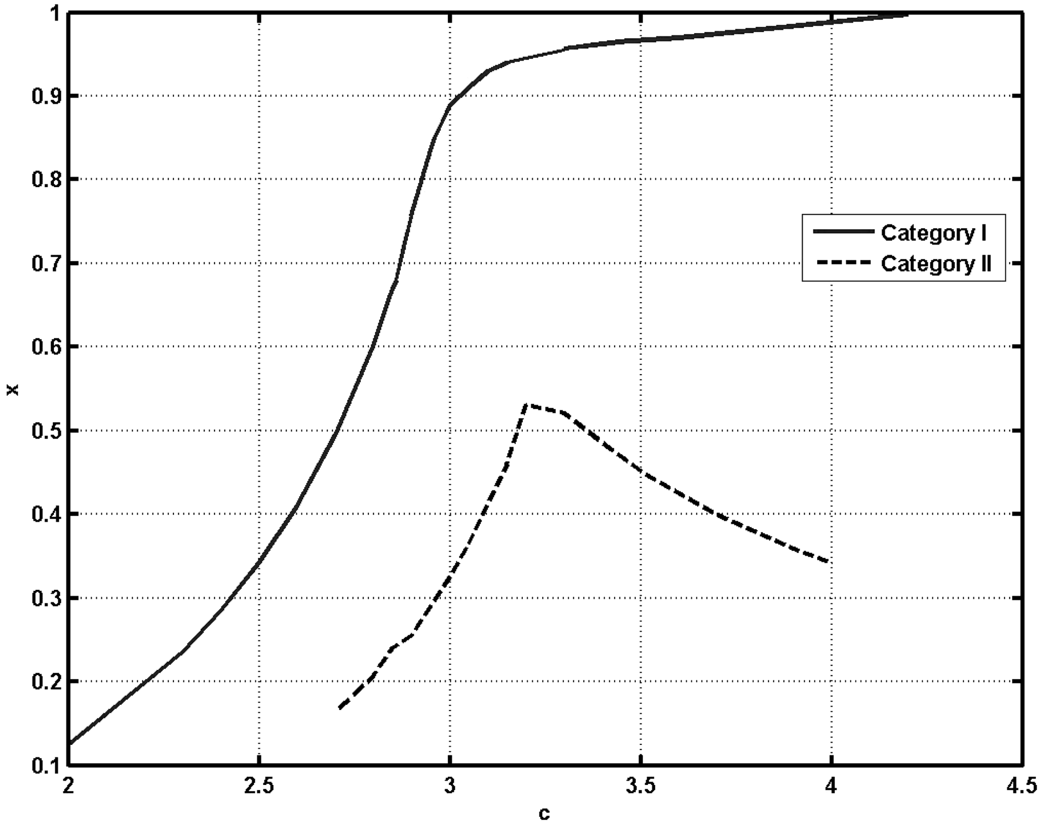

Fig. 7 Location of the periodic orbit as a function of Jacobi constant for the two categories of periodic orbits.

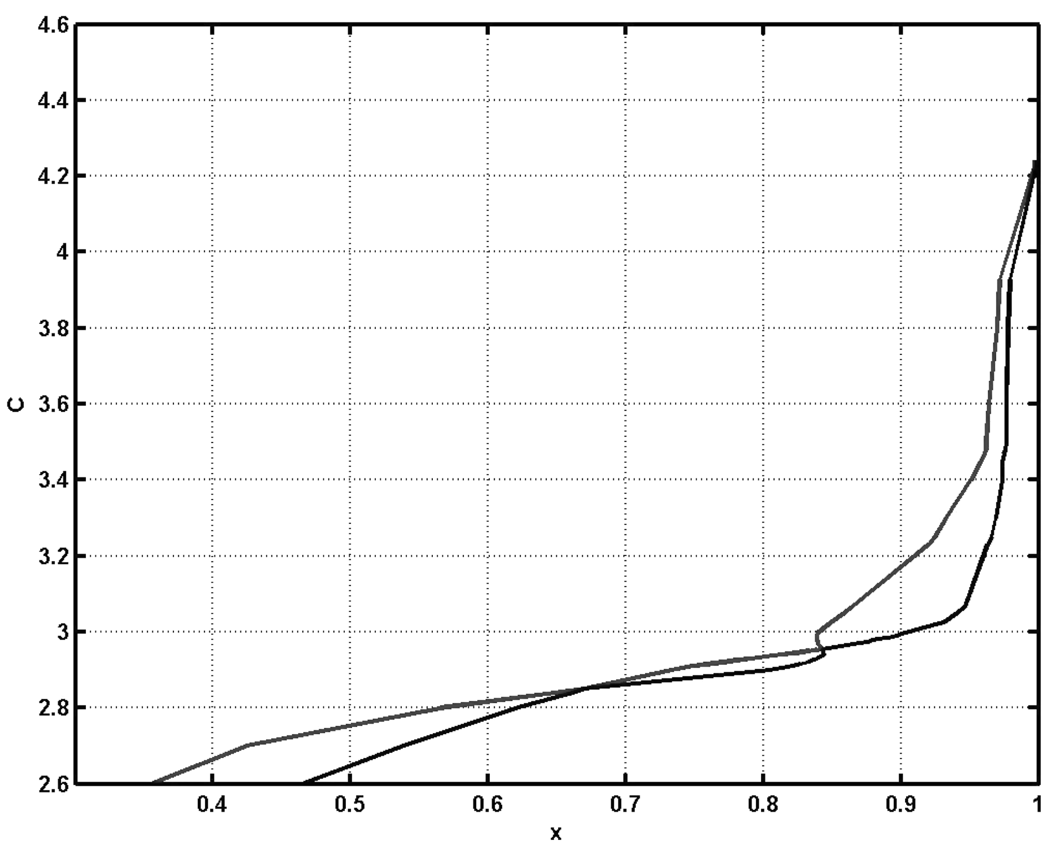

Fig. 8 Size of KAM tori as a function of Jacobi constant *C*. The upper line corresponds to the left most tip and the lower line corresponds to the rightmost tip of the island obtained by Poincaré surface of section, at the line of conjunction.

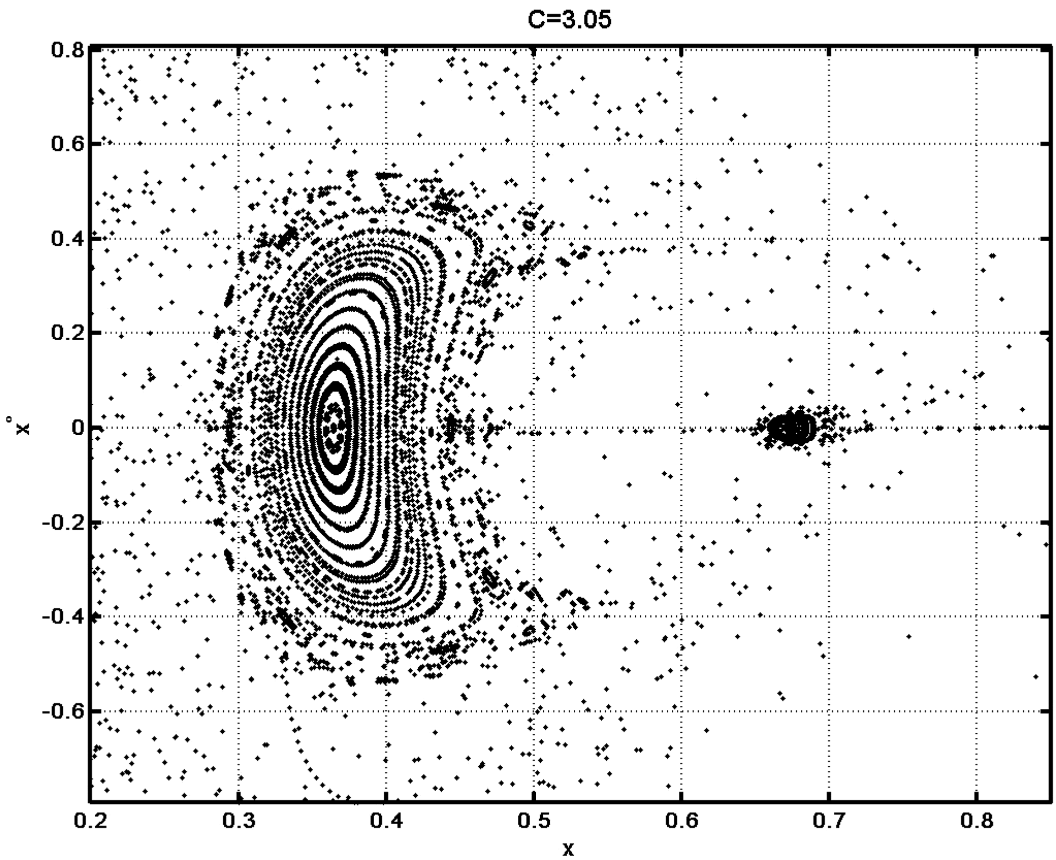

Fig. 9 Poincaré surface of section for Jacobi constant $C = 3.05$.

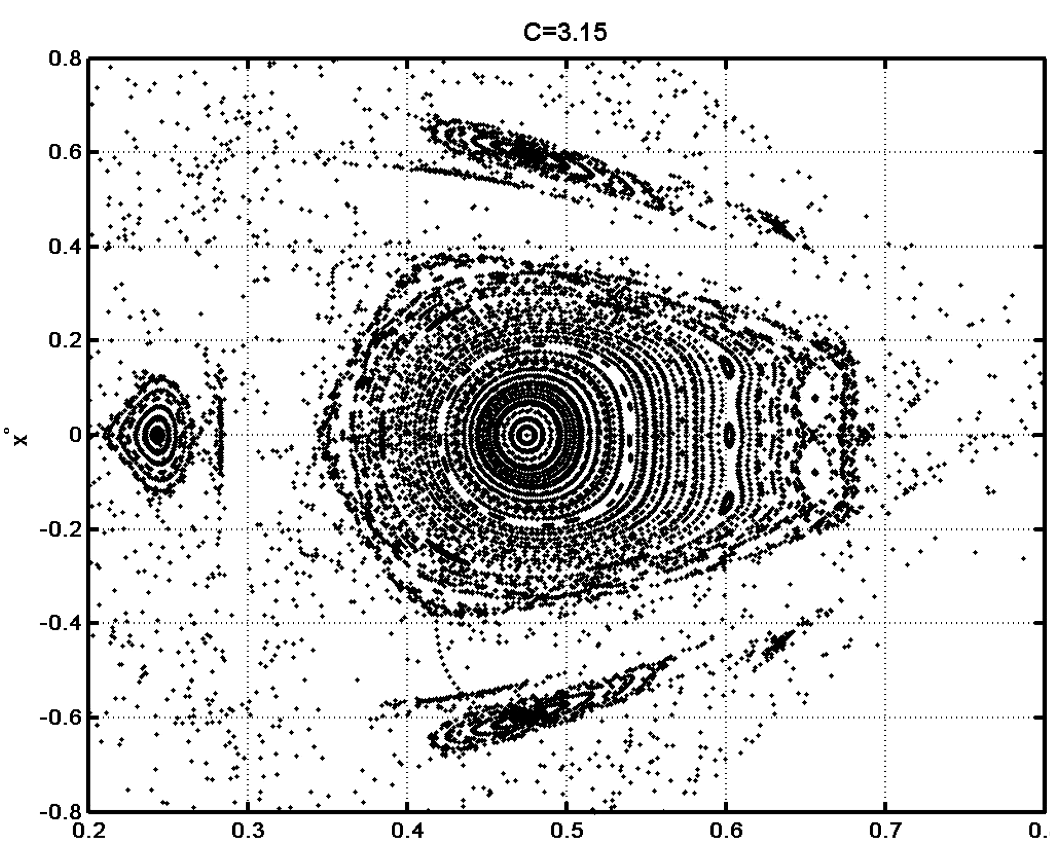

Fig. 10 Poincaré surface of section for Jacobi constant $C = 3.15$.

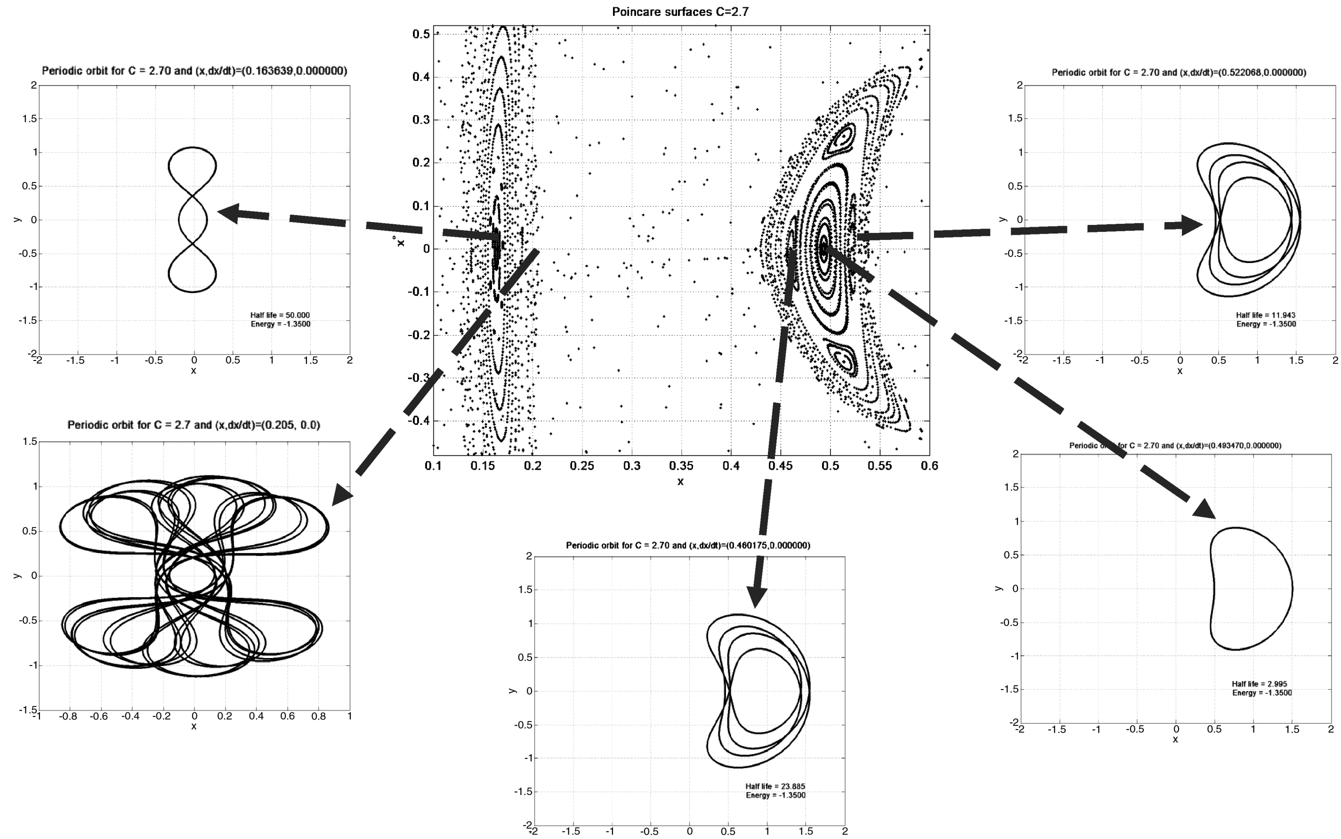

Fig. 11 Various orbits obtained from the Poincaré surface of section for $C = 2.7$.

results and figures do not match reasonably with Winter [16]. The differences may be attributed to the finer step size of $\Delta x = 0.001$ that we have used in our study. The step size used by Winter is 0.01. In the third stage, the size of the KAM tori increases up to $C = 3.066$ and then it decreases as C increases up to 4.24, where the periodic orbits are very close to the moon [17].

The disappearance of the region of stability is caused by the intersection of the central periodic orbit and the unstable periodic orbit lying at the three corners of the triangular stability region [20]. This transition at $C = 2.84895$ and 2.955 is shown in Figs. 1–6. This category of periodic orbits slowly traverses toward the moon. Figure 7 gives the location of the periodic orbit as a function of the Jacobi constant. These results were generated from the Poincaré surface of section, considering, for each Jacobi constant, the size of the largest island (quasi-periodic orbit) in the line of conjunction (values of x when $\dot{x} = 0$). Figure 8 provides the size of the KAM tori from $C = 2.6$ to 4.3.

Meanwhile, another set of islands is created. They attain peak stability, merge with other sets of islands, and mostly remain near Earth. Figure 7 clearly shows the evolution of category II periodic orbits at $C = 2.7$, whose stability keeps increasing up to $C = 3.05$. Meanwhile, another set appears at $C = 2.85$. Its stability increases and then merges with the bigger island. The merged orbit keeps gaining stability thereafter with increase in the value of C. Figures 9 and 10 give the representatives of this family.

One of the interesting surfaces of section is for $C = 2.7$, which has been expanded in Fig. 11. The center of the very first (segregated) island, which belongs to category II is at $x = 0.163639$ and $\dot{x} = 0.0$. With this initial condition when the system of equations of motion (1) and (2) is integrated, a stable periodic orbit around Earth is obtained. The central point of the second island is at $x = 0.460175$ and $\dot{x} = 0.0$, which yields a kidney-bean-shaped stable periodic orbit around the moon. Similarly shaped quasi-periodic orbits are obtained from the other points of the surface of section. It is clearly seen that as we move away from the center of the island, the periodic orbit slowly changes into a quasi-periodic orbit, preserving the shape.

Sensitivity to initial conditions was also analyzed in some of the cases. If the trajectory is started from the point at the lower end of the bigger island, with $x = 0.5919$ and $\dot{x} = -0.4224$, the particle no longer remains in the same region. Its trajectory moves toward the moon but maintains a distance. But if the initial condition is perturbed in the fifth decimal place and integrated, the trajectory moves and is captured at the moon near $t = 3.5$. Such sensitivity to initial conditions is suggestive of chaos.

V. Conclusions

We have used the technique of Poincaré surface of section to study a family of periodic and quasi-periodic orbits. Using this technique we have determined the location of the periodic orbits and their stability in terms of the maximum amplitude of oscillation. We have divided this family of orbits into two categories: namely, the category of islands near moon (category I) and those near the Earth (category II).

We found that the category I orbits slowly traverse toward the moon and that there is a kind of separatrix of two different types of quasi-periodic orbit around the periodic orbits at values of the Jacobi constant equal to 2.84895 and 2.955. The category II orbits remain mostly near Earth and their stability increases with increasing value of the Jacobi constant.

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References

[1] Poincaré, H., *Les Méthodes Nouvelles de la Mécanique Celeste*, Vol. 1, Gauthier-Villas, Paris, 1892, p. 82.

[2] Henon, M., "Exploration Numerique du Probleme Restreint: I," *Annales d'Astrophysique*, Vol. 28, Feb. 1965, pp. 499–511.

[3] Henon, M., "Exploration Numerique du Probleme Restreint: II," *Annales d'Astrophysique*, Vol. 28, Feb. 1965, pp. 992–1007.

[4] Henon, M., "Exploration Numerique du Probleme Restreint: III," *Bulletin Astronomique*, Ser. 3, Vol. 1, 1965, pp. 57–79.

[5] Henon, M., "Exploration Numerique du Probleme Restreint: IV," *Bulletin Astronomique*, Ser. 3, Vol. 2, 1965, pp. 49–66.

[6] Jefferys, W. H., "An Atlas of Surfaces of Section for the Restricted Problem of Three Bodies," Ser. II, Vol. 3, Dept. of Astronomy, Univ. of Texas at Austin, Austin, TX, p. 6.

[7] Smith, R. H., "The Onset of Chaotic Motion in the Restricted Problem of Three Bodies," Ph.D. Dissertation, Univ. of Texas at Austin, Austin, TX, 1991.

[8] Smith, R. H., and Szebehely, V., "The Onset of Chaotic Motion in the Restricted Problem of Three Bodies," *Celestial Mechanics and Dynamical Astronomy*, Vol. 56, No. 3, 1993, pp. 409–425. doi:10.1007/BF00691811

[9] Tuckness, D. G., "Position and Velocity Sensitivities at the Triangular Libration Points," *Celestial Mechanics and Dynamical Astronomy*, Vol. 61, No. 1, 1995, pp. 1–19. doi:10.1007/BF00051686

[10] Tuckness, D. G., "Time- and Phase-Space Stability Analysis of the Jupiter-Sun System," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 2, March–April 2005, pp. 355–359. doi:10.2514/1.2877

[11] Scott, C. J., and Spencer, D. B., "Stability Mapping of Distant Retrograde Orbits and Transport in the Circular Restricted Three-Body Problem," *Astrodynamics Specialist Conference and Exhibit*, AIAA Paper 2008-6431, Aug. 2008.

[12] Howell, K. C., and Kakoi, M., "Transfers Between the Earth-Moon and Sun-Earth Systems Using Manifolds and Transit Orbits," *Acta Astronautica*, Vol. 59, Nos. 1–5, 2006, pp. 367–380. doi:10.1016/j.actaastro.2006.02.010

[13] Demeyer, J., and Gurfil, P., "Transfer to Distant Retrograde Orbits Using Manifold Theory," *Journal of Guidance, Control, and Dynamics*, Vol. 30, No. 5, Sept.–Oct. 2007, pp. 1261–1267. doi:10.2514/1.24960

[14] Guilera, F. G., "On the dynamics of the Trojan asteroids," Ph.D. Thesis, Dept. de Mathematica Aplicada I Analisi, Universitat de Barcelona, Barcelona, 2003.

[15] Kolemen, E., Kasdin, N. J., and Gurfil, P., "Quasi-Periodic Orbits of the Restricted Three-Body Problem Made Easy," *New Trends in Astrodynamics and Applications III*, Vol. 886, American Inst. of Physics, Melville, NY, 2007, pp. 68–77.

[16] Winter, O. C., "The Stability Evolution of a Family of Simply Periodic Lunar Orbits," *Planetary and Space Science*, Vol. 48, No. 1, 2000, pp. 23–28. doi:10.1016/S0032-0633(99)00082-3

[17] Broucke, R. A., "Periodic Orbits in the Restricted Three-Body Problem with Earth-Moon Masses," Jet Propulsion Lab., TR 32-1168, Pasadena, CA, 1968.

[18] Szebehely, V., *Theory of Orbits*, Academic Press, New York, 1967, pp. 7–15.

[19] Sharma, R. K., Subba Rao, P. V., "Stationary Solutions and Their Characteristic Exponents in the Restricted Three-Body Problem When the More Massive Primary is an Oblate Spheroid," *Celestial Mechanics*, Vol. 13, No. 2, 1976, pp. 137–149. doi:10.1007/BF01232721

[20] Henon, M., "Numerical Exploration of the Restricted Problem, VI: Hill's Case: Non-Periodic Orbits," *Astronomy and Astrophysics*, Vol. 9, Nov. 1970, pp. 24–36.